

Art Galleries and Mobile Guards: Revisiting O’Rourke’s Proof

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Abstract

O’Rourke (1983) proved that every n -vertex polygon, with $n \geq 4$, can be guarded by $\lfloor \frac{n}{4} \rfloor$ edges or diagonals—a variant of Chvátal’s theorem for sufficiency of $\lfloor \frac{n}{3} \rfloor$ vertices. We present a short proof for a somewhat stronger result that allows us to impose some constraints on the guards. We prove that for every given subset V of vertices, the polygon can be guarded by $\lfloor \frac{n+2|V|}{4} \rfloor$ edges or diagonals that include at least one edge or diagonal incident to every vertex of V . This bound is the best achievable given the constraint for V . Our proof is by induction and suggests a simple linear-time algorithm after triangulating the polygon. The sufficiency of $\lfloor \frac{n}{4} \rfloor$ guards is a special case of the new result where V is the empty set.

1 Introduction

The problems of guarding polygons with vertices, edges, and diagonals have a rich background. They are usually referred to as the “art gallery problems”; see, for instance, the book by O’Rourke [8] and the survey by Shermer [9]. This type of problems has been initiated as early as 1973 by Victor Klee [4, 8, 9] and still is an active research area from both combinatorial and computational points of view, see e.g. [1, 2, 6, 10].

In 1983, O’Rourke [7] proved that $\lfloor \frac{n}{4} \rfloor$ mobile guards are always sufficient to *guard*—see the interior of—an n -vertex polygon with $n \geq 4$. A *mobile guard* is an edge or a diagonal of the polygon.¹ This bound is the best achievable as some polygons need these many mobile guards, see Figure 1(a) for an example, attributed to Toussaint [8, 9]. This is a variant of Chvátal’s theorem (1975) on the sufficiency of $\lfloor \frac{n}{3} \rfloor$ vertices to guard the polygon [4]. A short proof of Chvátal’s theorem was given by Fisk in 1987 [5] and it has been extended by Michael and Pinciu in 2016 [6].

To the best of our knowledge, O’Rourke’s original proof serves as the only proof of the sufficiency of $\lfloor \frac{n}{4} \rfloor$ mobile guards for almost 40 years. In this note we present an alternative proof for a more general result. The new proof is significantly shorter than the original proof, and allows us to impose some constraints on the guards. The trick is to find a suitable extension of the claim.

Let P be a simple polygon and G be a triangulation of P . To guard P it suffices to have one vertex of each triangular face of G on some mobile guard. We use the terms *edge* and *diagonal* to refer to the edges and diagonals of P . A *guard set* is a set of mobile guards that guard P .

O’Rourke’s Proof Technique. This proof is by induction, and it is mainly combinatorial. It involves careful edge contractions in G , which result in a somewhat different triangulation, without affecting the guarding process. To incorporate induction, it finds a diagonal d in G that cuts off 5, 6, 7, or 8 edges of P , and then applies the induction hypothesis on the remaining portion of P . The proof then involves thorough case analysis, crafted nicely, to patch the two parts together.

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¹O’Rourke uses the terms “mobile guard” and “diagonal guard” interchangeably.

The existence of d is implied by a useful result of Chvátal [4] that, for any $k \geq 2$, G has a diagonal that cuts off at least k and at most $2k - 2$ edges P . The original proof of Chvátal shows this for $k = 4$, O'Rourke applies it for $k = 5$, but it can be generalized to any $k \geq 2$.

New Proof Technique. We first find a suitable extension of the result that incorporates a stronger induction hypothesis. This allows us to cut a smaller portion of P that results in fewer cases with shorter proofs. We use a diagonal d that cuts off 3 or 4 edges of P . Such a diagonal exists by Chvátal's result for $k = 3$. In Lemma 1 we give an alternative existential proof, that is just as simple, but uses the dual of G .

2 The New Proof

In this section we prove the following theorem. The sufficiency of $\lfloor \frac{n}{4} \rfloor$ guards is a special case of the theorem where V is the empty set.

Theorem 1. *Let P be a polygon with $n \geq 4$ vertices and V be a subset of vertices. Then $\lfloor \frac{n+2|V|}{4} \rfloor$ mobile guards are sufficient to guard P such that each vertex in V is incident to a mobile guard.*

The bound in Theorem 1 is the best achievable given the constraint for V ; see Figure 1(b). It is well-known that the dual of any triangulation of P is a tree T of degree at most three. Moreover, each edge of T corresponds to a diagonal of P . Each leaf of T corresponds to an *ear* of P —a triangle formed by three consecutive vertices of P .

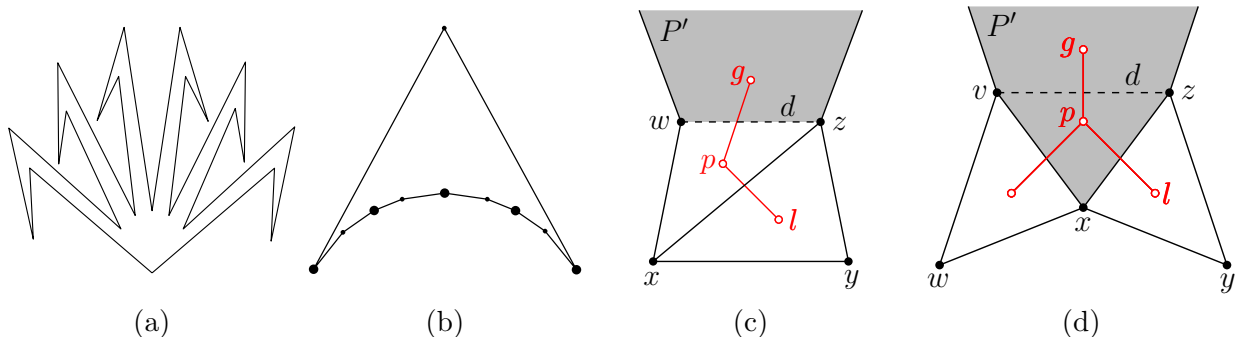


Figure 1: (a) A polygon that needs $\lfloor \frac{n}{4} \rfloor$ guards. (b) A polygon that needs $\lfloor \frac{n+2|V|}{4} \rfloor$ guards; bold vertices are in V . (c)-(d) Illustration of the proofs of Lemma 1 and Theorem 1; The tree T is in red, and the polygon P' is shaded.

Lemma 1. *Every triangulation of a polygon with $n \geq 6$ vertices has a diagonal that cuts off 3 edges of the polygon, or 4 edges of the polygon that are incident to two adjacent ears.*

Proof. Consider the dual tree T , and root it at a vertex of degree 1. Let l be a deepest leaf, p be its parent, and g be its grandparent. The node p has at most two children, which are leaves due to our choice of l . The diagonal of P that corresponds to the edge (g, p) cuts off 3 edges if p has one child or 4 edges if p has two children; see Figure 1(c) and 1(d). In the latter case the two children correspond to two adjacent ears. \square

Proof of Theorem 1. Let G be a triangulation of P . We will guard the triangular faces of G by choosing the edges of G (which are edges or diagonals of P) as our mobile guards. We prove by

induction on n that P can be guarded by $\lfloor \frac{n+2|V|}{4} \rfloor$ edges of G such that each vertex in V is incident to an edge in the guard set. The base cases are proved at the end. Now we proceed with the induction step. Take a diagonal d in G (as in Lemma 1) that cuts off 3 or 4 edges of P .

Case 1. The diagonal d cuts 3 edges say wx , xy , and yz . Then $d = wz$. Since G is a triangulation, $wy \in G$ or $xz \in G$. Without loss of generality assume that $xz \in G$. Let P' be the polygon with $n-2$ vertices that is obtained from P by cutting triangles xyz and wxz , as in Figure 1(c). Let G' be the graph obtained by removing vertices x, y and their incident edges from G . We set $V' = V \setminus \{x, y\}$ and consider the following cases.

- Case 1(a) ($x \notin V$ and $y \notin V$) If $z \notin V'$ then we add z to V' . Hence $|V'| \leq |V| + 1$. By the induction hypothesis we guard P' , with respect to G' and V' , by at most $\lfloor \frac{(n-2)+2(|V|+1)}{4} \rfloor = \lfloor \frac{n+2|V|}{4} \rfloor$ edges of G' . These mobile guards also guard triangles xyz and wxz , and hence P , because z is incident to a mobile guard.
- Case 1(b) ($x \in V$ and/or $y \in V$) Then $|V'| \leq |V| - 1$. By the induction hypothesis we guard P' , with respect to G' and V' , by at most $\lfloor \frac{(n-2)+2(|V|-1)}{4} \rfloor = \lfloor \frac{n+2|V|}{4} \rfloor - 1$ edges of G' . By adding the edge xy to these mobile guards we obtain a desired guard set for P .

Case 2. The diagonal d cuts 4 edges say vw , wx , xy , and yz . Then $d = vz$, and, by Lemma 1, vw and xy are two adjacent ears in P . Let P' be the polygon with $n-2$ vertices that is obtained from P by cutting these ears, as in Figure 1(d). Let G' be the graph obtained by removing vertices w, y and their incident edges from G . We set $V' = V \setminus \{w, y\}$ and consider three cases.

- Case 2(a) ($w \notin V$ and $y \notin V$) If $x \notin V'$ then we add x to V' . Hence $|V'| \leq |V| + 1$. By the hypothesis we guard P' , with respect to G' and V' , by at most $\lfloor \frac{(n-2)+2(|V|+1)}{4} \rfloor = \lfloor \frac{n+2|V|}{4} \rfloor$ edges of G' . These guards also guard P because x is incident to a guard.
- Case 2(b) ($w \in V$ or $y \in V$ but not both) Due to symmetry assume that $w \in V$, and hence $|V'| = |V| - 1$. We guard P' by at most $\lfloor \frac{(n-2)+2(|V|-1)}{4} \rfloor = \lfloor \frac{n+2|V|}{4} \rfloor - 1$ guards, and then add wx to obtain a desired guard set for P .
- Case 2(c) ($w \in V$ and $y \in V$) If $x \notin V'$ then we add x to V' . Hence $|V'| \leq |V| - 1$. Then we guard P' by a guard set of size at most $\lfloor \frac{(n-2)+2(|V|-1)}{4} \rfloor = \lfloor \frac{n+2|V|}{4} \rfloor - 1$. This guard set has at least one of the two edges of G' incident to x , say the edge xz . We remove xz from this set and add wx and yz . This gives a desired guard set for P (the triangles in G' that were guarded by xz are now guarded by wx or yz).

Base Cases. Each inductive step decreases n down by 2. Thus, for the base case it suffices to consider $n = 4, 5$. First assume that $n = 4$. Then any edge of P (which is also an edge of G) would guard P . If $|V| = 0$ then we guard P by an arbitrary edge. If $|V| = 1$ then we guard P by one edge incident to the only vertex of V . If $|V| = 2, 3, 4$ then we guard P by two opposite edges which are incident to all four vertices; in this case $2 = \lfloor \frac{4+2*2}{4} \rfloor \leq \lfloor \frac{4+2|V|}{4} \rfloor$.

Now assume that $n = 5$. Then G has two diagonals of P that are incident to the same vertex x . Hence any edge or diagonal incident to x , or any two arbitrary edges would guard P . If $|V| = 0$ then we guard P by an edge incident to x . If $|V| = 1$ then we guard P by an edge or diagonal that is incident to x and to the vertex of V . If $|V| = 2, 3$ then we guard P by two edges that are incident to all vertices of V , in this case $2 = \lfloor \frac{4+2*2}{4} \rfloor \leq \lfloor \frac{4+2|V|}{4} \rfloor$. If $|V| = 4, 5$ then we guard P by three edges incident to all five vertices, in this case $3 = \lfloor \frac{4+2*4}{4} \rfloor \leq \lfloor \frac{4+2|V|}{4} \rfloor$. This finishes the proof. \square

Algorithmic Implication Our proof of Theorem 1 suggests a linear-time algorithm for finding such a guard set. First we obtain an arbitrary triangulation G of P using Chazelle’s linear-time algorithm [3]. Then we maintain two sets D_3 and D_4 of diagonals of G that cut 3 or 4 edges, respectively, of the current polygon. We initialize these sets in linear time by checking, for each vertex v of P , at most 4 vertices that appear before v and at most 4 vertices that appear after v along P . In each induction step, after cutting P by a diagonal d we update D_3 and D_4 in constant time by checking at most 4 vertices that appear before or after each endpoint of d along P' .

Notice that, in view of Lemma 1, one can maintain D_3 and D_4 in linear time also by checking the leaves of the dual tree T iteratively.

3 Concluding Remarks

One can extend Theorem 1 further by imposing more constraints. In particular, if E and F are subsets of the edges of P , then one can show the sufficiency of $\lfloor \frac{n+3|E|+2|V|+|F|}{4} \rfloor$ mobile guards that include every edge of E and at least one endpoint of every edge of F . This can be proven by considering a few more subcases in our proof, however, due to our desire of having a shorter proof we leave this to the interested reader.

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